

THE INTEGRAL ON QUANTUM SUPER GROUPS OF TYPE $A_{r|s}$

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ABSTRACT. We compute the integral on matrix quantum (super) groups of type $A_{r|s}$ and derive from it the quantum analogue of (super) HCIZ integral.

INTRODUCTION

A formula for the integral on the group $U(n)$ was obtained by Itzykson and Zuber [3]. It can be given in the following form

$$\int_{U(n)} \text{tr}(MUNU^{-1})^k [dU] = \sum_{\lambda \in \mathcal{P}_k^n} \frac{d_\lambda}{r_\lambda} \Phi_\lambda(M) \Phi_\lambda(N) \quad (1)$$

for any hermitian matrices M and N . Φ_λ is the irreducible character of $U(n)$, corresponding to λ . If $\xi_1, \xi_2, \dots, \xi_n$ are the eigenvalues of M then $\Phi_\lambda(M) = s_\lambda(\xi_1, \xi_2, \dots, \xi_n)$, s_λ are the Schur functions. d_λ is the dimension of the irreducible module of the symmetric function \mathfrak{S}_n , r_λ is the dimension of the irreducible representation of $U(n)$, corresponding to partition λ . Explicitly, $d_\lambda = n! \prod_{x \in [\lambda]} h(x)^{-1}$, $r_\lambda = \prod_{x \in [\lambda]} c(x)h(x)^{-1}$, where $c(x)$ is the content, $h(x)$ is the hook-length of the box x in the diagram $[\lambda]$. This formula turns out to be a special case of a formula obtained by Harish-Chandra [11]. The integral on the left-hand side of (1) is therefore referred as Harish-Chandra-Itzykson-Zuber (HCIZ) integral. A super analogue of this formula was obtained by Alfaro, Medina and Urrutia [1, 2], it reads

$$\int_{U(m|n)} \text{str}(MUNU^{-1})^k [dU] = \sum_{\substack{\mu \in \mathcal{P}^m, \nu \in \mathcal{P}^n \\ |\mu| + |\nu| = k - mn}} \frac{k!}{|\mu|! |\nu|!} \frac{d_\mu d_\nu}{r_\mu r_\nu} \Phi_\lambda(M) \Phi_\lambda(N), \quad (2)$$

where $\lambda = (n^m) + \mu \cup \nu'$. The aim of this paper is to give a quantum analogue of the super HCIZ integral. In contrary to the method of Itzykson-Zuber and its super version developed by Alfaro *et. al.*, which is analytic, our method is purely algebraic. Thus, fist we want to give an algebraic definition of the integral.

It is well-known that an integral on a compact group is uniquely defined by its left (or right) invariance with respect to the group action. Thus, the integral can be considered as a linear functional on the function algebra of the group with an invariance property. Notice that the algebra of polynomial function on a compact (Lie) group is a commutative Hopf algebra. Being motivated by this fact, one can give the notion of integral on an arbitrary Hopf algebra, although, such an integral

does not always exist. Thus, by definition, a left integral on a Hopf algebra H is a linear form \int on H such that $(\text{id} \otimes \int)\Delta = \int$. Analogously, one has the notion of right integral. Generally, left and right integrals may differ. For general properties of integral on Hopf algebras, the reader is referred to [17].

Originally, the integral is defined only on compact groups. The algebraically defined integral exists however on any cosemisimple Hopf algebras and on finite dimensional Hopf algebras. In particular, Hopf algebras of functions on linear reductive groups possess integrals. This is explained by the fact that each linear reductive group possesses a compact form and the corresponding integral coincides with the analytically defined one on this compact form. Having this relationship in mind, we shall find in this work the integral on the function algebras on quantum (super) groups of type $A_{r-1|s-1}$, such a quantum group is understood to be a generalization of the quantum general linear super group $GL_q(r|s)$. It is defined in terms of a Hecke symmetry of birank (r, s) .

Our problem of finding integrals on the function algebra of quantum linear supergroup $GL_q(r|s)$ is thus motivated by the HCIZ integral. On the other hand, it is an interesting problem from the point of view of Hopf algebra theory. Integrals on Hopf algebras were studied by several authors since the pioneering work of Sweedler [17], see e.g. [16, 13, 5]. For finite dimensional Hopf algebra it is known that the integrals exist uniquely up to a scalar. However, only very few examples of infinite dimensional Hopf algebras with integrals, except cosemisimple Hopf algebras.

In the case of quantum groups of type A_r , which corresponds to Hecke symmetry with birank $(r, 0)$, the integral was computed in [9]. In this case, the function algebra is co-semisimple hence we know a priori the existence of the integral. In the general case, the function algebra is not co-semisimple. However, the formula for the integral in the former case suggests us an idea of finding an integral in the latter case.

Let R be a Hecke symmetry on a finite dimensional vector super space V of dimension d , and H_R the associate Hopf algebra of function on a quantum group of type $A_{r-1|s-1}$, where (r, s) is the birank of R , see Section 1. Thus, as an algebra, H_R is generated by $2d^2$ generators $z_i^j, t_i^j, 1 \leq i, j \leq d$. Using the commutation rule on H_R , one can show that an element of H_R can be represented as a linear combination of monomials on z_i^j, t_k^l of the form $Z_I^J T_K^L := z_{i_1}^{j_1} \cdots z_{i_p}^{j_p} t_{k_1}^{l_1} \cdots t_{k_q}^{l_q}$. From the linearity of the integral, we see that it is sufficient to find the integral on the set of monomials of the form $Z_I^J T_L^K$. Let us denote $|I|$ the cardinal number of the sequence I contents. It turns out that any integral should vanish on those monomial $Z_I^J T_K^L$ which has $|I| \neq |K|$.

The formula of the integral on $Z_I^J T_K^L$ with $|I| = |K|$ is based on an operator $P_n : V^{\otimes n} \longrightarrow V^{\otimes n}$, $n = |I| \neq |K|$. In Section 2 we show that the axiom for the integral is equivalent to certain conditions on P_n . In Section 3 we construct P_n . An advantage of our method comparing to the one of Itzykson-Zuber and Alfaro et.al. is that we are able to compute the integral at every monomial function, while their

method gives a formula of the integral only at certain trace-polynomials. In Section 4 we derive a quantum analogue of HCIZ integral from our integral formula. To do this we have to introduce the notion of character of H_R -comodules, the latter are understood to be rational representation of the quantum group. In the last section we discuss the orthogonality relation of simple H_R -comodules.

1. QUANTUM GROUPS ASSOCIATED TO HECKE SYMMETRIES

Let V be a super vector space over \mathbb{K} , a fixed field of characteristic zero. Fix a homogeneous basis x_1, x_2, \dots, x_d of V . We shall denote the parity of the basis element x_i by \hat{i} . An even operator R on $V \otimes V$ can be given by a matrix R_{ij}^{kl} : $R(x_i \otimes x_j) = x_k \otimes x_l R_{ij}^{kl}$. R is an even operator means that the matrix elements R_{kl}^{ij} are zero, except for those with $\hat{i} + \hat{j} = \hat{k} + \hat{l}$. R is called *Hecke symmetry* if the following conditions are satisfied:

- i) R satisfies the Yang-Baxter equation $R_1 R_2 R_1 = R_2 R_1 R_2$, $R_1 := R \otimes I$, $R_2 := I \otimes R$, I is the identity matrix of degree d .
- ii) R satisfies the Hecke equation $(R - q)(R + 1) = 0$ for some q which will be assumed *not to be a root of unity*.
- iii) There exists a matrix P_{ij}^{kl} such that $P_{jn}^{im} R_{ml}^{nk} = \delta_l^i \delta_j^k$. A matrix satisfying this condition is called *closed*.

The matrix bialgebra E_R and the matrix Hopf algebra H_R are define as follows. Let $\{z_j^i, t_j^i | 1 \leq i, j \leq d\}$ be a set of variables, $x_j^i = t_j^{\hat{i}} = \hat{i} + \hat{j}$. We define E_R as the quotient algebra of the free non-commutative algebra, generated by $\{z_j^i | 1 \leq i, j \leq d\}$, by the relations

$$(-1)^{\hat{s}(\hat{i}+\hat{p})} R_{ps}^{kl} z_i^p z_j^s = (-1)^{\hat{l}(\hat{q}+\hat{k})} z_q^k z_n^l R_{ij}^{qn}, \quad 1 \leq i, j, k, l \leq d. \quad (3)$$

Here, we use the convention of summing up by the indices that appear in both lower and upper places. And we define the algebra H_R as the quotient of the free non-commutative algebra generated by $\{z_j^i, t_j^i | 1 \leq i, j \leq d\}$, by the relations

$$(-1)^{\hat{s}(\hat{i}+\hat{p})} R_{ps}^{kl} z_i^p z_j^s = (-1)^{\hat{l}(\hat{q}+\hat{k})} z_q^k z_n^l R_{ij}^{qn}, \quad 1 \leq i, j, k, l \leq d, \quad (4)$$

$$(-1)^{\hat{j}(\hat{j}+\hat{k})} z_j^i t_k^j = (-1)^{\hat{l}(\hat{i}+\hat{i})} t_l^i z_k^l = \delta_k^i, \quad 1 \leq i, k \leq d. \quad (5)$$

The relations in (3) can be considered as the commuting rule for elements of E_R . For H_R , we have the following relations, which follow immediately form (4) and (5).

$$(-1)^{\hat{k}(\hat{i}+\hat{j})} R_{ql}^{pj} z_j^i t_k^l = (-1)^{\hat{m}(\hat{n}+\hat{p})} t_n^p z_q^m R_{mk}^{ni}, \quad (6)$$

$$(-1)^{\hat{s}(\hat{i}+\hat{p})} R_{ps}^{kl} t_j^s t_i^p = (-1)^{\hat{l}(\hat{q}+\hat{k})} t_n^l t_q^k R_{ij}^{qn}. \quad (7)$$

It is easy to show that E_R is a bialgebra, the coproduct on E_R and H_R is given by

$$\Delta(z_j^i) = z_k^i \otimes z_j^k, \quad \Delta(t_j^i) = t_j^k \otimes t_k^i.$$

H_R is, in fact, a Hopf algebra. The coproduct is given by

$$\Delta(z_j^i) = z_k^i \otimes z_j^k, \quad \Delta(t_j^i) = t_j^k \otimes t_k^i,$$

the antipode on H_R is given by

$$S(z_j^i) = (-1)^{\hat{j}(\hat{i}+\hat{j})} t_j^i, \quad S(t_j^i) = (-1)^{\hat{i}(\hat{i}+\hat{j})} C_k^i z_l^k C^{-1}_j,$$

where $C_j^i := P_{jl}^{il}$. To verify the axiom of Hopf algebra for H_R , we need the following relation

$$(-1)^{\hat{j}(\hat{i}+\hat{j})} z_k^l C_l^j t_j^i = C_k^i. \quad (8)$$

We also define $D_j^i := P_{lj}^{li}$. The matrices C and D play important roles in our work, they are called *reflection operators*. Using the Hecke equation we can show that (cf [9])

$$CD = DC = q^{-1} - (Q^{-1} - 1)\text{tr}(C).$$

Since we are working in the category of super vector spaces, the rule of sign effects on the coproduct. More precisely, the compatibility of product and coproduct of a super bialgebra reads

$$\Delta(a \otimes b) = \sum_{(a)(b)} (-1)^{\hat{a}_2 \hat{b}_1} a_1 b_1 \otimes a_2 b_2.$$

Therefore we have

$$\Delta(z_{j_1}^{i_1} z_{j_2}^{i_2} \cdots z_{j_n}^{i_n}) = \text{sign}(I, K) \text{sign}(K, J) \text{sign}(I, J) z_{k_1}^{i_1} z_{k_2}^{i_2} \cdots z_{k_n}^{i_n} \otimes z_{j_1}^{k_1} z_{j_2}^{k_2} \cdots z_{j_n}^{k_n},$$

where $I := (i_1, i_2, \dots, i_n)$ and so on. $\text{sign}(I, J)$ is given recurrently by

$$\text{sign}(i, j) = 1, \quad \text{sign}(Ii, Ji) = (-1)^{\hat{i}(|\hat{I}|+|\hat{J}|)} \text{sign}(I, J),$$

$\hat{I} := (\hat{i}_1, \hat{i}_2, \dots, \hat{i}_n)$, $|\hat{I}|$ denotes the sum of its terms. Hence, for convenience, we denote

$$Z_J^I = Z^{\otimes n I} : \text{sign}(I, J) z_{j_1}^{i_1} z_{j_2}^{i_2} \cdots z_{j_n}^{i_n}.$$

Throughout this paper we shall always use this notation. Then we have

$$\Delta(Z_J^I) = Z_K^I \otimes Z_L^K, \quad \Delta(T_J^I) = T_L^K \otimes T_K^I,$$

and $S(Z_J^I) = (-1)^{|\hat{J}|(|\hat{I}|+|\hat{J}|)} T_{J'}^{I'}$. Notice that

$$\text{sign}(K', L') = (-1)^{|\hat{K}_1|(\hat{k}_n + \hat{j}_n)} \text{sign}(K'_1, L'_1),$$

where for $K = (k_1, k_2, \dots, k_n)$, $K_1 := (k_1, k_2, \dots, k_{n-1})$ and $K' := (k_n, k_{n-1}, \dots, k_1)$.

The Hecke algebra of type A , $\mathcal{H}_{n,q}$ is generated by elements T_i , $1 \leq i \leq n-1$, subject to the relations

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i^2 = (q-1)T_i + q, \quad i = 1, \dots, n-2.$$

To each element w of the symmetric group \mathfrak{S}_n of permutations of the sets $\{1, 2, \dots, n\}$, one can associate in a canonical way an element T_w of $\mathcal{H}_n = \mathcal{H}_{n,q}$, in particular, $T_1 = 1$, $T_{(i,i+1)} = T_i$. The set $\{T_w | w \in \mathfrak{S}_n\}$ form a \mathbb{K} basis for \mathcal{H}_n .

R induces an action of the Hecke algebra $\mathcal{H}_n = \mathcal{H}_{q,n}$ on the tensor powers $V^{\otimes n}$ of V , $\rho_n(T_i) = R_i := \text{id}_V^{i-1} \otimes R \otimes \text{id}_V^{n-i-1}$. We shall therefore use the notation $R_w := \rho(T_w)$. On the other hand, E_R coacts on V by $\delta(x_i) = x_j \otimes z_i^j$. Since E_R is a bialgebra, it coacts on $V^{\otimes n}$ by means of the product. The double centralizer theorem [8] asserts that these two actions are centralizers of each other in $\text{End}_{\mathbb{K}}(V^{\otimes n})$ [8]. Hence, the algebra $\text{End}^{E_R}(V^{\otimes n})$ is a factor algebra of \mathcal{H}_n and $(E_R^n)^* \cong \text{End}_{\mathcal{H}_n}(V^{\otimes n})$. Let us denote $\text{End}^{E_R}(V^{\otimes n})$ by $\overline{\mathcal{H}_n}$. Since \mathcal{H}_n is semi-simple, provided q is not a root of unity, the algebras $(E_R^n)^*$ and $\overline{\mathcal{H}_n}$ are semi-simple, too.

The double centralizer theorem also implies that a simple E_R -comodule is the image of the operator induced by a primitive idempotent of \mathcal{H}_n and, conversely, each primitive idempotent of \mathcal{H}_n induces an E_R comodule which is either zero or simple. On the other hand, irreducible representations of \mathcal{H}_n are parameterized by partitions of n . Thus, up to conjugation, primitive idempotents of \mathcal{H}_n are parameterized by partitions of n , too. Note that by the semisimplicity, $\overline{\mathcal{H}_n}$ is also a subalgebra of \mathcal{H}_n .

The primitive idempotents $x_n := \sum_w T_w/[n]_q!$ and $y_n := \sum_w (-q)^{-l(w)} T_w/[n]_{1/q}!$ induce the symmetrizer and anti-symmetrizer operators on $V^{\otimes n}$. Let $S_n := \text{Im} \rho_n(x_n)$ and $\Lambda_n := \text{Im} \rho_n(y_n)$. Then one can show that $S := \bigoplus_{n=0}^{\infty} S_n$ and $\Lambda := \bigoplus_{n=0}^{\infty} \Lambda_n$ are algebras. They are called symmetric and exterior tensor algebras on the corresponding quantum space.

By definition, the Poincaré series $P_{\Lambda}(t)$ of Λ is $\sum_{n=0}^{\infty} t^n \dim_{\mathbb{K}}(\Lambda_n)$. It is proved that this series is a rational function having negative roots and positive poles. Let r be the number of its roots and s be the number of its poles. As a consequence, simple E_R -comodules are parameterized by hook-partitions from $\Gamma_n^{rs} := \{\lambda \vdash n \mid \lambda_{r+1} \leq s\}$ [7]. Therefore, in the algebra $\overline{\mathcal{H}_n}$ primitive idempotents are parameterized by hook-partitions from $\Gamma_n^{r,s}$, too.

(r, s) is called the birank of R . Our main assumption on R is that

$$\text{tr}(C) = -[s - r]_q, \quad (9)$$

C is the reflection operator introduced above. It can be proved that $\text{tr}(C)$ should have the form $-[x]_q$ for some integer x in the interval $[-s, r]$ [6]. The above equation holds for any known Hecke symmetry. It is conjectured that it holds for all Hecke symmetry.

Simple H_R -comodules are much more complicated. The problem of classifying all its simple comodules is still open.

The Hopf algebra H_R (resp. the bialgebra E_R) is called the (function algebra on) a quantum group (resp. quantum semigroup) of type $A_{r-1|s-1}$.

The following are two main examples of Hecke symmetries. The Drinfeld-Jimbo's R -matrix of type A_{r-1} [12], for $1 \leq i, j, k, l \leq r$, $p^2 = q$,

$$R_{rij}^{kl} := \begin{cases} q & \text{if } i = j = k = l \\ q - 1 & \text{if } k = i > j = l \\ p & \text{if } k = j \neq i = l \\ 0 & \text{otherwise.} \end{cases}$$

Assume that all parameters are even, the Hopf algebra associated to R_r is called the (function algebra on) standard quantum general linear group $\mathrm{GL}_q(r)$. R_r is a one-parameter deformation of the permuting operator. The super version of this operator was given by Manin [15]. Assume that the variable x_i , $i \leq r$ are even and the rest s variables are odd. Define, for $1 \leq i, j, k, l \leq r + s$, $p^2 = q$,

$$R_{r|s}^{kl} := \begin{cases} q & \text{if } i = j = k = l, \hat{i} = 0 \\ -1 & \text{if } i = j = k = l, \hat{i} = 1 \\ q - 1 & \text{if } k = i < j = l \\ (-1)^{\hat{i}\hat{j}} p & \text{if } k = j \neq i = l \\ 0 & \text{otherwise.} \end{cases}$$

$R_{r|s}$ is a deformation of the permuting operator in super symmetry. The associated Hopf algebra is called the (function algebra on) standard quantum general linear super group $\mathrm{GL}_q(r|s)$.

R_r has the birank $(r, 0)$. $R_{r|s}$ has the birank (r, s) .

2. THE INTEGRAL ON H_R

Recall that by definition, a left integral on a Hopf algebra H over a field k is a non trivial lineal functional $\int : H \longrightarrow k$ with the invariance property:

$$\int(a) = \sum_{(a)} a_1 \otimes a_2. \quad (10)$$

Since we are considering super algebra, we shall also require that the integral is even, that means the value of an integral at an odd element of H_R is zero. It easy to see that (10) is equivalent to

$$\sum_{(b)} \int(aS(b_1))b_2 = \sum_{(a)} a_1 \int(a_1S(b)). \quad (11)$$

From the definition of H_R , an arbitrary element of H_R can be represented by as a linear combination of monomials in z_i^j and t_k^l . On the other hand, using the relation on H_R and the axiom (iii) of R , we can represent a monomial like $t_j^i z_k^l$ as a linear combination of monomials of the form $z_p^q t_r^s$, i.e., we can interchange the order of z' and t' s in a tensor product. Namely, we have, according to (7),

$$(-1)^{\hat{l}(\hat{i}+\hat{j})} t_i^j z_k^l = (-1)^{\hat{r}(\hat{p}+\hat{q})} R_{ks}^{jp} z_p^q t_r^s P_{qi}^{rl}. \quad (12)$$

Thus, by using the rule (12), we can represent any element of H_R as a linear combination of monomials of the form

$$Z_I^J T_K^L := z_{i_1}^{j_1} \cdots z_{i_p}^{j_p} \cdots t_{k_1}^{l_1} \cdots t_{k_p}^{l_p}.$$

Therefore, it is sufficient to compute the integral at such monomials. Let us denote, for $n = |I|$,

$$I_{nIK}^{JL} := (-1)^{|\hat{K}|(|\hat{K}|+|\hat{J}|)} \int (Z_I^L T_{K'}^{J'})$$

where, as defined in the previous section, K' is the same sequence as K but in the reverse order. We have the following conditions on I_n :

- (i) I_n should be invariant with respect to the relation with thin z 's and t 's, given in (4) and (7), respectively, that is, for all $i, j = 1, 2, \dots, n-1$

$$(R_i \otimes R_j) I_n = I_n (R_j \otimes R_i).$$

- (ii) when we contract I_n with respect to the relation (5), (8), we should get I_{n-1} , more precisely,

$$\begin{aligned} \delta_{j_n}^{i_n} I_{nI_1 i_n K}^{J_1 j_n L} &= I_{n-1 I_1 K_1}^{J_1 L_1} \delta_{kn}^{l_n} \\ C_{l_n}^{k_n} I_{nIK k_n}^{JL_1 l_n} &= I_{n-1 I_1 p K_1}^{J_1 j_n L_1} C_{j_n}^{i_n}. \end{aligned}$$

- (iii) I_n should respect the rule (11), which reads

$$I_{nIM}^{JL} Z_K^M = Z_N^L I_{nIK}^{JN}.$$

This condition is, in fact, sufficient for an integral on H_R . For, assume we have a collection of matrices I_n , satisfying the conditions (i-iii) above, then we can extend it linearly on the whole H_R . The only ambiguity that may occur is that, there may be more than one way of leading an element of H_R to a linear combination of monomials of the form $Z_I^J T_K^L$. However, the Yang-Baxter equation on R ensures that different ways of using rule (12) give us the same result, up to relations in z 's and t 's, respectively.

We thus reduced the problem to finding a family of matrices I_n satisfying conditions (i-iii). Our next claim is that I_n can be found in the following way

$$I_n = \sum_{w \in \mathfrak{S}_n} (-q)^{-l(w)} (P_n C^{\otimes n} R_{w^{-1}}) \otimes R_w \quad (13)$$

where $R_w = \rho(T_w)$ as in Section 1, C is the reflection operator introduced in Section 1 and P_n is a certain operator on $V^{\otimes n}$. More precisely, we have

Lemma 2.1. *Assume that the operator $P_n \in \overline{\mathcal{H}_n} = \text{End}^{H_R}(V^{\otimes n}) \subset \text{End}^k(V^{\otimes n})$ is in the center of $\overline{\mathcal{H}_n}$ and satisfies the condition*

$$(P_{n-1} \otimes \text{id}_V) = P_n(L_n + \text{tr}(C))$$

where L_n are the Murphy operators: $L_1 = 0$,

$$L_n = \sum_{i=1}^{n-1} q^{-i} R_{(n-i,n)}, \quad n \geq 2,$$

$(n-i, n)$ is the inversion that changes places of $n-i$ and n . Then the matrices I_n given (13) satisfy the conditions (i-iii).

Proof. The conditions (i) and (iii) can be easily verified. In fact, (i) is equivalent to the equations

$$\begin{aligned} \sum_{w \in \mathfrak{S}_n} (-q)^{-l(w)} (R_i P_n C^{\otimes n} R_{w^{-1}} \otimes R_w) = \\ \sum_{w \in \mathfrak{S}_n} (-q)^{-l(w)} (P_n C^{\otimes n} R_{w^{-1}}) \otimes (R_w R_i), \end{aligned}$$

for $i = 1, 2, \dots, n-1$. By assumption, P_n commutes with all R_i . On the other hand, using the Yang-Baxter equation we can also show that $C^{\otimes n}$ commutes with all R_i . Therefore the equation above implies from the following

$$\sum_{w \in \mathfrak{S}_n} (-q)^{-l(w)} (R_i R_{w^{-1}}) \otimes R_w = \sum_{w \in \mathfrak{S}_n} (-q)^{-l(w)} (R_{w^{-1}}) \otimes (R_w R_i)$$

which can be easily verified using the Hecke equation for R . The verification of (iii) is straightforward, it does not involve P_n and $C^{\otimes n}$ but rather a direct consequence of relations (4).

The harder part is to verify (ii). Here we use the condition

$$(P_{n-1} \otimes \text{id}_V)(L_n + \text{tr}(C)) = P_n.$$

In fact, the operator L_n comes into play by the following equality.

It is known that each element T_w of \mathcal{H}_n can be expressed in the form $T_w = T_k \cdots T_{n-1} T_{w_1}$ for some $w_1 \in \mathfrak{S}_{n-1}$, where \mathfrak{S}_{n-1} is the subgroup of \mathfrak{S}_n , fixing n , i.e. \mathfrak{S}_{n-1} permutes only $\{1, 2, \dots, n-1\}$. We define a linear map $\mathcal{H}_n \rightarrow \mathcal{H}_{n-1}$ setting

$$h_n(T_w) = \begin{cases} T_k \cdots T_{n-2} T_{w_1} & \text{if } k \leq n-2 \\ \text{tr}(C) T_{w_1} & \text{if } k = n-1. \end{cases}$$

Then we have an identity in \mathcal{H}_n

$$\sum_{w \in \mathfrak{S}_n} (-q)^{-l(w)} T_{w^{-1}} \otimes h_n(T_w) = \sum_{u \in \mathfrak{S}_{n-1}} (-q)^{-l(w)} (L_n + \text{tr}(C)) T_{u^{-1}} \otimes T_u \quad (14)$$

here we identify \mathcal{H}_{n-1} with the subalgebra of \mathcal{H}_n generated by $T_u, u \in \mathfrak{S}_{n-1}$. For the proof the reader is referred to [9]. In fact, we can replace $\text{tr}(C)$ by any element of \mathbb{K} , but the crucial point here is that

$$C_{j_n}^{i_n} R_{w I_1 i_n}^{J_1 j_n} = h_n(R_w I_1^J).$$

Here, we specialize h_n on the algebra $\overline{\mathcal{H}_n}$.

When we replace T_w by R_w in (14) we obtain immediately the second equation in (ii). More work is needed for the first equation of (ii). Interested reader is again referred to [9] for detail.

Thus, we reduced the problem of finding an integral on H_R to constructing operators $P_n \in \overline{\mathcal{H}_n}$ satisfying certain conditions. This step is motivated by the construction of the integral for H_R when the birank of R is $(r, 0)$ in [9]. The

essential step is to construct P_n . The main difficulty here is that, unlike the case of birank $(r, 0)$, the operators $L_n + \text{tr}(C)$ are not invertible in $\overline{\mathcal{H}_n}$, so that we cannot follow the old way to set $P_n = (P_{n-1} \otimes \text{id}_V)(L_n + \text{tr}(C))^{-1}$.

3. THE CONSTRUCTION OF P_n

We want to construct operators $P_n \in \overline{\mathcal{H}_n}$ with the property

$$P_n(L_n + \text{tr}(C))^{-1} = P_{n-1} \otimes \text{id}_V.$$

Originally the Murphy operators were introduced by Dipper and James [4] to describe a full set of mutually orthogonal primitive idempotents of the algebra \mathcal{H}_n : the set

$$E_{t_i(\lambda)} = \prod_{\substack{1 \leq m \leq n \\ |k| \leq m-1 \\ k \neq c_{t_i(\lambda)}(m)}} \frac{L_m - [k]_q}{[c_{t_i(\lambda)}(m)]_q - [k]_q}, \quad 1 \leq i \leq d_\lambda, \quad \lambda \in \mathcal{P}_n,$$

where $c_{t_i(\lambda)}(m)$ is the content of m in the standard tableau $t_i(\lambda)$, is a full set of mutually orthogonal primitive idempotents of \mathcal{H}_n . The idempotents $E_{t_i(\lambda)}$, $i = 1, 2, \dots, d_\lambda$ belong to the same block that corresponds to λ , their sum $\sum_{1 \leq i \leq d_\lambda} E_{t_i(\lambda)} = F_\lambda$ – the minimal central idempotent corresponding to λ .

L_m satisfy the equation

$$\prod_{k=-m-1}^{m+1} (L_m - [k]_q) = 0.$$

Therefore, for $1 \leq m \leq n$,

$$L_m E_{t_i(\lambda)} = E_{t_i(\lambda)} L_m = c_{t_i(\lambda)}(m) E_{t_i(\lambda)}. \quad (15)$$

We define

$$p_\lambda := \prod_{x \in [\lambda] \setminus [(s^r)]} \frac{q^{r-s}}{[c_\lambda(x) + r - s]_q},$$

and

$$P_n := \sum_{\substack{\lambda \in \Omega_n^{r,s} \\ 1 \leq i \leq d_\lambda}} p_\lambda E_{t_i(\lambda)} = \sum_{\lambda \in \Omega_n^{r,s}} p_\lambda F_\lambda. \quad (16)$$

Recall that P_n are defined in the algebra $\overline{\mathcal{H}_n} \cong \text{End}^{E_R}(V^{\otimes n})$, which is the factor algebra of \mathcal{H}_n by the two-sided ideal generated by minimal central idempotents corresponding to partitions from $\mathcal{P}_n \setminus \Gamma_n^{r,s}$. Fixing an embedding $\overline{\mathcal{H}_n} \hookrightarrow \overline{\mathcal{H}_{n+1}}$, $\overline{\mathcal{H}_n} \ni W \longmapsto W \otimes \text{id}_V \in \overline{\mathcal{H}_{n+1}}$, we identify $\overline{\mathcal{H}_n}$ with a subalgebra of $\overline{\mathcal{H}_{n+1}}$.

Lemma 3.1. *The operators P_n are central in $\overline{\mathcal{H}_n}$ and satisfy the equation (in $\overline{\mathcal{H}_{n+1}}$)*

$$P_{n+1}(L_{n+1} - [s - r]_q) = P_n.$$

Proof. The operator P_n is obviously central, for it is a sum of central elements.

We check the equation above. First, notice that, if $\lambda \in \Omega_{n+1}^{r,s}$ and $t_i(\lambda)$ is a standard λ -tableau, then the node of $[\lambda]$, containing $n+1$, is removable, i.e., having removed it we still have a standard tableau. The tableau $t_i(\lambda)$ is called essential if this node is not the node (r, s) , otherwise, it is called non-essential. A tableau $t_i(\lambda)$ is essential iff the tableau, obtained from it by removing the node containing $n+1$ is again a γ -tableau with $\gamma \in \Omega_n^{r,s}$.

Observe, that if $t_i(\lambda)$ is non-essential, then $c_{t_i(\lambda)} = s - r$, hence

$$E_{t_i(\lambda)}(L_{n+1} - [s - r]_q) = 0,$$

by virtue of Equation (15). We reshuffle the terms of P_{n+1} in groups as follows

$$P_{n+1} = \sum_{\substack{\gamma \in \Omega_n^{r,s} \\ 1 \leq i \leq d_\gamma}} \sum_{\substack{\lambda \in \Omega_{n+1}^{r,s} \\ t(\lambda) \supset t_i(\gamma)}} p_\lambda E_{t(\lambda)} + \sum_{\substack{t(\lambda) \text{ is} \\ \text{not essential}}} p_\lambda E_{t(\lambda)}.$$

That is, we pick up into group for each $t_i(\gamma)$, $\gamma \in \Omega_n^{r,s}$, those standard tableaux $t(\lambda)$, that contain $t_i(\gamma)$ as a subtableau. The above observation implies that the second sum in the right-hand side of the above equation multiplied by $L_{n+1} - [s - r]_q$ vanishes. Thus, it is sufficient to prove, for a fixed $t_i(\gamma)$, $\gamma \in \Omega_n^{r,s}$,

$$\sum_{\substack{\lambda \in \Omega_{n+1}^{r,s} \\ t(\lambda) \supset t_i(\gamma)}} p_\lambda E_{t(\lambda)} (L_{n+1} - [s - r]_q) = q_\gamma E_{t_i(\gamma)}.$$

We have $(L_{n+1} - [s - r]_q) E_{t(\lambda)} = [c_{t(\lambda)}(n+1)]_q - [s - r]_q$ and $p_\lambda(c_\lambda(x) - [s - r]_q) = p_\gamma$ whenever $[\gamma]$ is obtained from $[\lambda]$ by removing the node x . Since, for any two standard tableaux $t(\gamma)$ and $t(\lambda)$ with $\gamma \subset \lambda$ as above, the number $n+1$ should lie in the node x , for which $[\lambda] \setminus [x] = [\gamma]$, we deduce that the equation to be proved is equivalent to

$$\sum_{\substack{\lambda \in \Omega_{n+1}^{r,s} \\ t(\lambda) \supset t_i(\gamma)}} E_{t(\lambda)} = E_{t_i(\gamma)}. \quad (17)$$

Since $\prod_{k=-n-1}^{n+1} (L_{n+1} - [k]_q) = 0$,

$$\sum_{m=-n-1}^{n+1} \prod_{\substack{k=-n-1, \\ k \neq m}}^{n+1} \frac{L_{n+1} - [k]_q}{[m]_q - [k]_q} = 1.$$

Therefore

$$E_{t_i(\gamma)} = \sum_{m=-n-1}^{n+1} E_{t_i(\gamma)} \prod_{\substack{k=-n-1, \\ k \neq m}}^{n+1} \frac{L_{n+1} - [k]_q}{[m]_q - [k]_q}.$$

Remember that we are working in the algebra $\overline{\mathcal{H}_{n+1}}$, in which $E_\lambda \neq 0$ iff $\lambda \in \Gamma_{n+1}^{r,s}$. Each term on the right-hand side of the above equation is either zero or a primitive idempotent of the form $E_{t(\lambda)}$ with $t(\lambda)$ containing $t_i(\gamma)$ as a subtableau. Since the left-hand side of (17) contains all primitive idempotents in $\overline{\mathcal{H}_{n+1}}$ that correspond

to standard tableaux containing $t_i(\gamma)$ as a subtableau, the equation (17) follows. Lemma 3.1 is therefore proved. ■

As a colorrary of Lemmas 2.1 and 3.1, we have

Theorem 3.2. *The Hopf algebra H_R associated to a Hecke symmetry R , which satisfies the condition (9), possesses an integral, which is uniquely determined up to a scalar multiple. Let (r, s) be the birank of R . Then an integral can be given as follows: if $l(I) = l(K) < rs$, $\int(Z_I^J T_K^L) = 0$, if $l(I) = l(K) = n \geq rs$,*

$$\int(Z_I^J T_{K'}^{L'}) = (-1)^{|\hat{K}|(|\hat{K}|+|\hat{L}|)} \sum q^{-l(w)} (P_n C^{\otimes n} R_{w^{-1}})_I^L R_{wK}^J, \quad (18)$$

$$(K' = (k_n, k_{n-1}, \dots, k_1)).$$

Notice that since the operators C, P, R_w are all even, the integral vanishes unless $|I| = |\hat{L}|, |K| = |\hat{J}|$.

There is a symmetric bilinear form on the Hecke algera \mathcal{H}_n , given by $(T_u, T_v) = q^{l(u)} \delta_{v^{-1}}^u$. With respect to this bilinear form, $\{R_w, w \in \mathfrak{S}_n\}$ and $\{q^{-l(w)} R_{w^{-1}}\}$ are dual bases. Thus, if $\{E_\lambda^{ij}, \lambda \vdash n, 1 \leq i, j \leq d_\lambda\}$ is a basisof \mathcal{H}_n with the following properties

1. $\{E_\lambda^{ij}, \leq i, j \leq d_\lambda\}$ is a basis of the block in \mathcal{H}_n , corresponding to λ ;
2. $E_\lambda^{ij} E_\mu^{kl} = \delta_\lambda^u \delta_k^j E_\lambda^{il}$.

then, using standard argument we can easily show

$$\sum_{w \in \mathfrak{S}_n} q^{-l(w)} R_{w^{-1}} R_w = \sum_{\substack{\lambda \vdash n \\ 1 \leq i, j \leq d_\lambda}} \frac{1}{k_\lambda} E_\lambda^{ij} \otimes E_\lambda^{ji}, \quad (19)$$

where $k_\lambda = (E_\lambda^{ii}, E_\lambda^{ii})$, (\cdot, \cdot) is the mentioned above bilinear form. k_λ can be computed explicitly (cf. [10])

$$k_\lambda = q^{n(\lambda)} \prod_{x \in [\lambda]} [h(x)]_q^{-1},$$

where $n(\lambda) = \sum_i \lambda_i(i-1)$, $h(x)$ is the hook length of x in the diagram $[\lambda]$: $h_\lambda(x) = \lambda_i + \lambda'_j - i - j + 1$, where (i, j) is the coordinate of x in the diagram $[\lambda]$.

The formula (18) can therefore be rewritten as follows:

$$\int(Z_I^J T_{K'}^{L'}) = (-1)^{|\hat{K}|(|\hat{K}|+|\hat{L}|)} \sum_{\substack{\lambda \vdash n \\ 1 \leq i, j \leq d_\lambda}} \frac{1}{k_\lambda} E_\lambda^{ijL} E_\lambda^{jiJ}, \quad (20)$$

4. CHARACTERS OF H_R AND QUANTUM ANALOGUE OF SUPER HCIZ INTEGRAL

For a coribbon Hopf algebra, one associate to each finite-dimensional H -comodule M a trace map $\Phi : \text{End}_{\mathbb{K}}(M) \longrightarrow H$, satisfying the following conditions, for $f, g \in$

$\text{End}_{\mathbb{K}}(M)$, $h \in \text{End}_{\mathbb{K}}(N)$,

$$\Phi(f \circ g) = \Phi(g \circ f) \quad (21)$$

$$\Phi(f + g) = \Phi(g) + \Phi(g) \quad (22)$$

$$\Phi(f \otimes g) = \Phi(f)\Phi(g). \quad (23)$$

The reader is referred to [10] for the explicit construction of the trace map Φ . Note that the ribbon structure on H is to ensure the condition (23) for Φ . The character of comodule M is defined to be $\Phi(\text{id}_M) \in H$.

Our Hopf algebra H_R is a coribbon Hopf algebra (cf. [6]). We state without proof the following lemma. The reader can also consider this lemma as the definition of the characters of H_R -comodules M_λ and their duals.

For each partition $\lambda \in \Gamma_n^{r,s}$, M_λ denotes the corresponding simple E_R -comodule. Since the map $E_R \longrightarrow H_R$ is injective (cf. [9]), M_λ is simple H_R -comodule, too. M_λ is isomorphic to $\text{Im} \rho(E_\lambda)$ of a primitive idempotent E_λ . Therefore $\Phi(M_\lambda) = \Phi(E_\lambda)$. Thus, setting $D_j^i = r(z_l^i, S(z_j^l))$, we have

Lemma 4.1. *The characters of M_λ and M_λ^* , $\lambda \vdash n$ are given by*

$$\begin{aligned} S_\lambda &:= \Phi(M_\lambda) = q^{n(r-s+1)/2} \text{tr}(D^{\otimes n} E_\lambda Z^{\otimes n}), \\ S_{-\lambda} &:= \Phi(M_\lambda^*) = q^{n(r-s+1)/2} \text{tr}(C^{\otimes n} E_\lambda \overline{T^{\otimes n}}), \end{aligned}$$

where $\overline{T^{\otimes n}}_J^I := T_{J'}^{I'}$.

Remember that in the definition of $Z^{\otimes n}$, the signs are also inserted.

The proof of this lemma does not differ from the one for the non-super case given in [10].

Let K be an algebra. A K -point of a (super) bialgebra B is an algebra homomorphism $\mathcal{A} : B \longrightarrow K$. Let \mathcal{A} be a K -point of the bialgebra E_R . Then the entries of A commute by the same rule as the entries of Z . We define $S_\lambda(A) := \mathcal{A}(S_\lambda)$. In other words, $S_\lambda(A)$ is S_λ computed at $Z = A$. We are now ready to formulate a formula to compute the quantum super HCIZ integral.

Theorem 4.2. *Let M be a K -point of E_R and N be a K -point of $E'_R = S(E_R)$ (i.e., the subbialgebra of H_R , generated by T). Assume that entries of M and N commute and that they commute with the entries of Z and T . Then*

$$\int \text{tr}(D^{\otimes n} M^{\otimes n} Z^{\otimes n} \overline{N^{\otimes n} T^{\otimes n}}) = q^{-n(r-s+1)} \sum_{\lambda \in \Omega_n^{r,s}} \frac{d_\lambda p_\lambda}{k_\lambda} S_\lambda(M) S_{-\lambda}(N).$$

Proof.

$$\begin{aligned}
& \int \text{tr}(D^{\otimes n} M^{\otimes n} Z^{\otimes n} \overline{N^{\otimes n} T^{\otimes n}}) \\
&= \sum_{w \in \mathfrak{S}_n} q^{-l(w)} \text{tr}(P_n C^{\otimes n} R_{w^{-1}} \overline{N^{\otimes n}}) \cdot \text{tr}(R_w D^{\otimes n} M^{\otimes n}) \\
&= \sum_{\substack{1 \leq i, j \leq d_\lambda \\ \lambda \vdash n}} k_\lambda^{-1} \text{tr}(P_n E_\lambda^{ij} C^{\otimes n} \overline{N^{\otimes n}}) \cdot \text{tr}(E_\lambda^{ij} D^{\otimes n} M^{\otimes n}) \\
&= \sum_{\substack{1 \leq i, j \leq d_\lambda \\ \lambda \vdash n}} q^{-n(r-s+1)} k_\lambda^{-1} p_\lambda \Phi(E_\lambda^{ij*})(N) \cdot \Phi(E_\lambda^{ji})(M) \\
&= \sum_{\lambda \vdash n} q^{-n(r-s+1)} d_\lambda k_\lambda^{-1} p_\lambda S_{-\lambda}(N) S_\lambda(M).
\end{aligned}$$

Since $p_\lambda = 0$ whenever $\lambda \notin \Omega_n^{r,s}$, we obtain the desired equality. ■

Example. Let us consider the case of standard quantum general linear super group $GL_q(r|s)$, determined in terms of the symmetry $R_{r|s}$ given Section 1. In this case, any diagonal matrix with commuting entries is a point of E_R . Thus, assume that M and N are diagonal matrix with entries commuting each other and with the entries of Z and T , $M = \text{diag}(m_1, m_2, \dots, m_{r+s})$, $N = \text{diag}(n_1, n_2, \dots, n_{r+s})$. Then, we have

$$S_{(n)} = \sum_{k=0}^n h_{n-k}(qm_1, \dots, q^r, m_r) e_k(-q^r m_{r+1}, \dots, -q^{r-s+1} m_{r+s}),$$

h_k, e_k are the k^{th} complete and elementar symmetric functions in r and s variables, respectively. Hence

$$S_\lambda(M) = s_\lambda(qm_1, q^2 m_2, \dots, q^r m_r / -q^r m_{r+1}, \dots, -q^{r-s+1} m_{r+s}),$$

s_λ are the Hook-Schur functions in $r+s$ variables (cf. [14], Ex. I.3.23). Therefore, if $\lambda \in \Omega_n^{r,s}$, thus, $\lambda = (r^s) + \mu \cup \nu'$, $\mu \in \mathcal{P}^r, \nu \in \mathcal{P}^s$, we have [loc.cit.]

$$S_\lambda(M) = (-1)^{|\nu|} \prod_{i=1, j=1}^{r, s} (q^i m_i - q^{r-j+1} m_{r+j}) s_\mu(\{q^i m_i\}) s_\nu(\{q^{r+1-i} m_{r+i}\}),$$

where s_μ (resp. s_ν) are the Schur functions in r variables (resp. s variables). Analogously, we have

$$S_{-\lambda}(N) = (-1)^{|\nu|} \prod_{i=1, j=1}^{r, s} (q^{r-s-i+1} n_i - q^{j-s} n_{r+j}) s_\mu(\{q^{r-s+1-i} n_i\}) s_\nu(\{q^{i-s} n_{r+i}\}).$$

The quantum super HCIZ is then given by, ($n \geq rs$),

$$\begin{aligned} & \int \text{tr}(C^{\otimes n} N^{\otimes n} Z^{\otimes n} M^{\otimes n} T^{\otimes n}) = \\ & \sum_{\substack{\mu \in \mathcal{P}^r, \nu \in \mathcal{P}^s, \\ |\mu| + |\nu| = n - rs \\ \lambda = (s^r) + \mu \cap \nu'}} q^{r|\nu| - s|\mu|} p_\mu p_\nu^- \frac{d_\lambda}{k_\lambda} \prod_{i=1}^{r,s} (q^i m_i - q^{r-j+1} m_{r+j}) (q^{r-s-i+1} n_i - q^{j-s} n_{r+j}) \\ & \times s_\mu(\{q^i m_i\}) s_\nu(\{q^{r+1-i} m_{r+i}\}) s_\mu(\{q^{r-s+1-i} n_i\}) s_\nu(\{q^{i-s} n_{r+i}\}).. \end{aligned}$$

5. THE ORTHOGONAL RELATIONS

We are now interested in the orthogonal relations. Let M_λ, M_μ be two simple comodules corresponding to partitions λ and μ of n . Let $M_\lambda = \text{Im } E_\lambda$, $M_\mu = \text{Im } E_\mu$. Then, using Theorem 3.2, we have

$$\begin{aligned} (\Phi(M_\lambda), \Phi(M_\mu)) &= \int (\Phi(M_\lambda) \Phi(M_\mu^*)) \\ &= \sum q^{-l(w)} \text{tr}(P_n C^{\otimes n} E_\lambda R_{w^{-1}} E_\mu R_w) \\ &= \delta_\lambda^\mu \delta_{\Omega^{r,s}}^\lambda \frac{p_\lambda}{k_\lambda} \text{tr}(C^{\otimes n} E_\lambda) \\ &= 0. \end{aligned}$$

Here $\delta_{\Omega^{r,s}}^\lambda$ indicates, whether λ belongs to $\Omega^{r,s}$, it is zero if $\lambda \notin \Omega^{r,s}$ and 1 otherwise. On the other hand, for λ form $\Omega^{r,s}$, $\text{tr}(C^{\otimes n} E_\lambda) = 0$, for it is the quantum rank of M_λ .

We see that the scalar product above cannot be used to define the orthogonal relations. Following [1] we compute the integral $\int (Z_\lambda Z_{-\mu})$, where Z_λ is a coefficient matrix of the simple comodule M_λ , i.e., Z_λ is the multiplicative matrix corresponding to certain basis of M_λ .

Let us fix a primitive idempotent E_λ , $\lambda \vdash n$, and set $M_\lambda = \text{Im } (E_\lambda)$. To choose a basis of M_λ we proceed as follows. First notice that there exists a one-one correspondence between the set of bases of M_λ and the set of maximal operators P_b^a in $\text{End}_{\mathbb{K}}(M_\lambda)$ with property $P_b^a P_d^c = \delta_b^c P_d^a$. Further, since E_λ is an idempotent, $\text{End}_{\mathbb{K}}(M_\lambda) = E_\lambda \text{End}_{\mathbb{K}}(V^{\otimes n}) E_\lambda$. Therefore, instead of fixing a basis of M_λ we can equivalently choose a maximum set of operators $P_b^a = E_\lambda Q_b^a E_\lambda$, for some $Q_b^a \in \text{End}(V^{\otimes n})$, satisfying $P_b^a P_d^c = \delta_b^c P_d^a$.

Let C_λ be the restriction of $C^{\otimes n}$ on M_λ , which is an invariant space of $C^{\otimes n}$) and $C_{\lambda b}^a$ be the matrix element of C_λ with respect to the basis give by P_b^a , $1 \leq a, b \leq m_\lambda$, m_λ is the dimension of M_λ . Then we have

$$C_{\lambda b}^a = \text{tr}(C^{\otimes n} P_b^a).$$

Analogously, the multiplicative matrix, corresponding the basis determined by P_b^a , is given by $Z_{\lambda n}^a = \text{tr}(P_b^a Z^{\otimes n})$. For the dual comodule we have $Z_{-\lambda b}^a = \text{tr}(P_b^a \overline{T}^{\otimes n})$.

Now we can find the integral

$$\begin{aligned} \int (Z_{\lambda b}^a Z_{-\lambda d}^c) &= \int \text{tr}(P_b^a Z^{\otimes n}) \text{tr}(P_d^c \overline{T^{\otimes n}}) \\ &= k_\lambda^{-1} p_\lambda \text{tr}(C^{\otimes n} E_\lambda^{ij} P_b^a E_\lambda^{ji} P_d^c) \\ &= k_\lambda^{-1} p_\lambda \delta_b^c \text{tr}(C^{\otimes n} P_d^a) \\ &= k_\lambda^{-1} p_\lambda \delta_b^c C_{\lambda d}^a, \end{aligned}$$

for we can choose E_λ^{ij} such that $E_\lambda = E_\lambda^{ii}$ for some i . Thus, we have proved

Proposition 5.1. *Let M_λ be the simple comodule of H_R , corresponding to partition λ and e_1, e_2, \dots , be its basis. Let Z_λ be the corresponding multiplicative matrix and $Z_{-\lambda}$ be the multiplicative matrix corresponding to the dual basis on M_λ^* . Let C_λ be the restriction of the operator $C^{\otimes n}$ on M_λ and $C_{\lambda b}^a$ be its matrix elements with respect to the basis above. Then we have the following orthogonal-type relations:*

$$\int (Z_{\lambda b}^a Z_{-\lambda d}^c) = k_\lambda^{-1} p_\lambda \delta_b^c C_{\lambda d}^a, \quad (24)$$

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